THE ISODIAMETRIC PROBLEM AND OTHER INEQUALITIES IN THE CONSTANT CURVATURE 2-SPACES

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Abstract. In this paper we prove several new inequalities for centrally symmetric convex bodies in the 2-dimensional spaces of constant curvature κ, which have their analog in the plane. Thus, when κ tends to 0, the classical planar inequalities will be obtained. For instance, we get the relation between the perimeter and the diameter of a symmetric convex body (Rosenthal-Szasz inequality) which, together with the well-known spherical/hyperbolic isoperimetric inequality, allows to solve the isodiametric problem. The analogs to other classical planar relations are also proved.

1. Introduction

If K is a planar convex body, i.e., a compact convex set in $\mathbb{R}^2$, with area $\mathrm{A}(K)$ and diameter $\mathrm{D}(K)$, the well-known *isodiametric inequality* states that

$$\pi\mathrm{D}(K)^2 \geq 4\mathrm{A}(K), \tag{1}$$

with equality if and only if K is a circle. The isodiametric inequality can be obtained as a consequence of the famous *isoperimetric inequality*,

$$\mathrm{p}(K)^2 \geq 4\pi\mathrm{A}(K), \tag{2}$$

and the classical *Rosenthal-Szasz's theorem*, namely,

$$\mathrm{p}(K) \leq \pi\mathrm{D}(K); \tag{3}$$

here $\mathrm{p}(K)$ denotes the perimeter of K. In (2) equality holds if and only if K is a circle, whereas for (3) all constant width sets verify the equality.

For detailed information on these classical inequalities we refer to [3, § 10].

The isoperimetric problem has its analog on the sphere $\mathbb{S}^2_\kappa$ and the hyperbolic plane $\mathbb{H}^2_\kappa$ with curvature $\kappa \gtrless 0$: Bernstein [2] (respectively, Schmidt [13]) proved that if K is the region bounded by a convex curve on $\mathbb{S}^2_\kappa$ (respectively, $\mathbb{H}^2_\kappa$), then

$$\mathrm{p}(K)^2 \geq 4\pi\mathrm{A}(K) - \kappa\mathrm{A}(K)^2, \tag{4}$$

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with equality only for the geodesic discs (see also, e.g., [12, p. 324]). However, just a few classical inequalities of the Euclidean plane have been translated into the sphere and the hyperbolic space (for instance, Jung's inequality, see [4], or Bonnesen's inequality, see [10]). For other problems in the hyperbolic plane/space we refer, for instance, to [1, 5, 6, 7, 8].

The aim of this paper is to consider other classical inequalities of planar convex bodies, looking for their analog on the constant curvature spaces under symmetry assumptions. For instance, in the case of the sphere we get the following spherical isodiametric inequality:

Theorem 1.1. *Let K be a centrally symmetric convex body in $\mathbb{S}^2_\kappa$. Then*

$$\frac{4\pi^2}{\kappa}\sin^2\left(\sqrt{\kappa}\frac{\mathrm{D}(K)}{2}\right) \geq 4\pi\mathrm{A}(K) - \kappa\mathrm{A}(K)^2, \tag{5}$$

and equality holds if and only if K is a geodesic disc.

As in the case of the classical (planar) isodiametric inequality, (5) will be obtained as a direct consequence of the spherical isoperimetric inequality (4) and an analog to Rosenthal-Szasz theorem (3) for the sphere; namely, we prove the following result:

Theorem 1.2. *Let K be a centrally symmetric convex body in $\mathbb{S}^2_\kappa$. Then*

$$\mathrm{p}(K) \leq \frac{2\pi}{\sqrt{\kappa}}\sin\left(\sqrt{\kappa}\frac{\mathrm{D}(K)}{2}\right). \tag{6}$$

and equality holds if and only if K is a geodesic disc.

At this point we observe that if K is not centrally symmetric, then inequality (6) is not true, as an octant of the sphere shows easily. We also notice that in the two above results, when κ tends to 0 we obtain the classical isodiametric inequality (1) and the Rosenthal-Szasz result (3), respectively.

Inequalities (5) and (6) provide upper bounds for the area and the perimeter of a centrally symmetric spherical convex body, respectively. We also wonder, analogously to the planar case, whether they can be bounded from below. We give a positive answer to this question, involving the minimal width $\omega(K)$ of the convex body (see Section 4 for the definition).

Theorem 1.3. *Let K be a centrally symmetric convex body in $\mathbb{S}^2_\kappa$. Then*

$$\begin{aligned} \mathrm{p}(K) &\geq \frac{2\pi}{\sqrt{\kappa}}\sin\left(\sqrt{\kappa}\,\frac{\omega(K)}{2}\right), \\ \mathrm{A}(K) &\geq \frac{2\pi}{\kappa}\left[1-\cos\left(\sqrt{\kappa}\,\frac{\omega(K)}{2}\right)\right]. \end{aligned} \tag{7}$$

Equality holds in both inequalities if and only if K is a geodesic disc.

Theorems 1.1 and 1.2 are proved in Section 3, as well as other new inequalities (e.g. a spherical Bonnesen-type isodiametric inequality for centrally symmetric convex bodies, see Proposition 3.3). In Section 4 the results involving the minimal width, namely, Theorem 1.3 and other relations, are obtained. First, Section 2 is devoted to introduce the notation, definitions and some preliminary results which will be needed along the proofs.

In the case of the hyperbolic plane $\mathbb{H}^2_\kappa$, since the proofs of the corresponding results are analogous to the ones in the sphere, for the sake of brevity we just state the inequalities for $\mathbb{H}^2_\kappa$ in an appendix at the end of the paper.

2. Notation for the case of the sphere and preliminaries

Let $\mathbb{S}^2_\kappa$ be the 2-dimensional sphere of curvature $\kappa > 0$, i.e., with radius $1/\sqrt{\kappa}$. A subset $K \subsetneq \mathbb{S}^2_\kappa$ is said to be *convex* (in $\mathbb{S}^2_\kappa$) if it satisfies the following two properties:

- K is contained in an open hemisphere;
- for any $p, q \in K$, there exists a unique (arc length parametrized) geodesic segment, i.e., segment of great circle, joining p and q, which is contained in K.

We observe that uniqueness comes from the assumption that K is contained in an open hemisphere. From now on we will represent by γ_p^q the geodesic starting at p and passing through q, and for the sake of brevity, we will write $\overline{\gamma}_p^q$ to denote the (geodesic) segment of γ_p^q between p and q.

The set of all convex bodies, i.e., compact convex sets with non-empty interior, in the sphere $\mathbb{S}^2_\kappa$ will be denoted by $\mathcal{K}(\mathbb{S}^2_\kappa)$, and for $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$, we denote by $\mathrm{A}(K)$ and $\mathrm{p}(K)$ its area (Lebesgue measure) and perimeter (length of the boundary curve), respectively. We notice that for any $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$, $A(K) > 0$. Finally, $\mathrm{bd}\, K$ and $\mathrm{int}\, K$ will represent the boundary and the interior (in $\mathbb{S}^2_\kappa$) of K.

The intrinsic distance $d : K \times K \longrightarrow \big[0, \mathrm{D}(K)\big]$ is given by $d(p,q) = \mathrm{L}(\overline{\gamma}_p^q)$, i.e., the length of the geodesic segment $\overline{\gamma}_p^q$.

As usual in the literature, for $p \in \mathbb{S}^2_\kappa$ we represent the exponential map at p by $\exp_p : T_p\mathbb{S}^2_\kappa \longrightarrow \mathbb{S}^2_\kappa$, and for any $\mathrm{v} \in T_p\mathbb{S}^2_\kappa$, we write $\gamma_{p,\mathrm{v}}(t)$ to denote the geodesic passing through p with velocity vector v, i.e., $\gamma_{p,\mathrm{v}}(t) = \exp_p(t\mathrm{v})$, $t \geq 0$. The geodesic disc centered at p and with radius $r \geq 0$, namely,

$$D(p,r) = \exp_p\big(\{\mathrm{v} \in T_p\mathbb{S}^2_\kappa : |\mathrm{v}| \leq r\}\big),$$

coincides with the closed ball $B_d(p,r)$ in the intrinsic distance (see [9, Proposition 7.2.6]). Its perimeter and area take the values (see, e.g., [14, p. 85])

$$\begin{aligned} \mathrm{p}\big(D(p,r)\big) &= \frac{2\pi}{\sqrt{\kappa}} \sin\big(\sqrt{\kappa}\, r\big) \quad \text{and} \\ \mathrm{A}\big(D(p,r)\big) &= \frac{2\pi}{\kappa}\big[1 - \cos\big(\sqrt{\kappa}\, r\big)\big]. \end{aligned} \tag{8}$$

Clearly, $\lim_{\kappa\to 0} \mathrm{p}\big(D(p,r)\big) = 2\pi r$ and $\lim_{\kappa\to 0} \mathrm{A}\big(D(p,r)\big) = \pi r^2$, i.e., the perimeter and the area of a planar disc of radius r.

For $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$, the diameter $\mathrm{D}(K)$ is defined as the maximum intrinsic distance between two points of K, its circumradius $\mathrm{R}(K)$ is the greatest lower bound of all radii R such that K is contained in a geodesic disc of radius R, and the inradius $\mathrm{r}(K)$ is the least upper bound of all radii r such that K contains a geodesic disc of radius r.

2.1. **On the monotonicity of the area and the perimeter.** On the one hand, it is a well-known fact that the Lebesgue measure in the sphere (indeed any positive measure) is a monotonous functional, i.e., if $K, L \in \mathcal{K}(\mathbb{S}^2_\kappa)$ with $K \subseteq L$, then $\mathrm{A}(K) \leq \mathrm{A}(L)$.

On the other hand, in [10, Proposition 1.3], the monotonicity of the perimeter in $\mathcal{K}(\mathbb{S}^2_\kappa)$ was shown:

Lemma 2.1 ([10, Proposition 1.3]). *Let $K, L \in \mathcal{K}(\mathbb{S}^2_\kappa)$ such that $K \subseteq L$. Then $\mathrm{p}(K) \leq \mathrm{p}(L)$.*

The proof of this fact is based on the so-called *Principal Kinematic Formula* for $\mathbb{S}^2_\kappa$ (see e.g. [12, p. 321]). Next we provide a characterization of the equality cases, which will be needed for the proofs of the equality cases in the forthcoming results.

Lemma 2.2. *Let $K, L \in \mathcal{K}(\mathbb{S}^2_\kappa)$, $K \subseteq L$.*

i) *If $\mathrm{A}(K) = \mathrm{A}(L)$ then $K = L$.*
ii) *If $\mathrm{p}(K) = \mathrm{p}(L)$ then $K = L$.*

Proof. In order to prove i), we assume that $K \subsetneq L$. Then, on the one hand, there exist $p \in (\operatorname{int} L) \cap (L \backslash K)$, and hence $r > 0$ small enough such that $D(p, r) \subseteq L \backslash K$. Indeed, if $p \in L \backslash K$ and $p \notin \operatorname{int} L$, it would be $p \in \operatorname{bd} L$; then since K, L are compact and convex, the (spherical) convex hull (i.e., the smallest convex body in the sphere containing the set) $\operatorname{conv}\bigl(\{p\} \cup K\bigr) \subseteq L$ and $\operatorname{conv}\bigl(\{p\} \cup K\bigr) \backslash K$ would contain interior points of L.

Therefore, by the monotonicity of the area,

$$\mathrm{A}(L \backslash K) \geq \mathrm{A}\bigl(D(p, r)\bigr) > 0.$$

On the other hand, since $L = K \dot{\cup} (L \backslash K)$, the additivity of the area functional yields $\mathrm{A}(L) = \mathrm{A}(K) + \mathrm{A}(L \backslash K)$, and using the assumption $\mathrm{A}(L) = \mathrm{A}(K)$ we get $\mathrm{A}(L \backslash K) = 0$, a contradiction.

Next we show ii). Without loss of generality, and for the sake of brevity, we fix $\kappa = 1$, and we assume that $\mathrm{p}(K) = \mathrm{p}(L)$ and $K \subsetneq L$. Then there exists a point $p \in (\operatorname{bd} L) \backslash K$. Moreover, since K is compact, there exists $q \in \operatorname{bd} K$ such that $d(p, q) = \min\bigl\{d(p, r) : r \in \operatorname{bd} K\bigr\}$. Let $C = \operatorname{pos} K$ be the positive hull in $\mathbb{R}^3$ of K and let H be the (unique) supporting plane in $\mathbb{R}^3$ of C at q (see Figure 1). Denoting by H^+ the half-space determined by H and containing C, and taking $Q = L \cap H^+$, it is clear that $K \subseteq Q \subsetneq L$. Moreover, since H is a plane through the origin, $Q \cap H = \overline{\gamma}_{q_1}^{q_2}$ is a geodesic segment with, say, end points q_1, q_2. We observe, on the one hand, that since L is convex, then q_1, q_2 cannot be antipodal points, and thus the curve segment α in $\operatorname{bd} L$ joining q_1 and q_2 (and passing through p) is not a geodesic segment (see Figure 1).

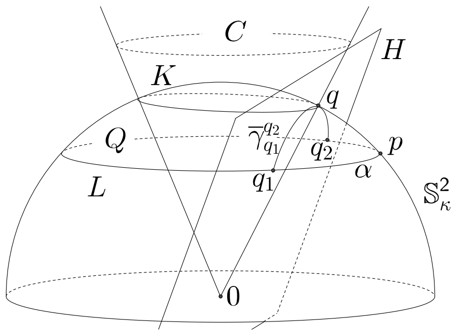

FIGURE 1. The set Q and the curves joining q_1, q_2.

Therefore, $\mathrm{L}(\overline{\gamma}_{q_1}^{q_2}) < \mathrm{L}(\alpha)$, because the intrinsic distance between two points in the same open hemisphere is minimized by geodesic segments, and since $p \notin H^+$, we finally get

$$\mathrm{p}(Q) < \mathrm{p}(L). \tag{9}$$

On the other hand, by Lemma 2.1 we have that

$$\mathrm{p}(K) \leq \mathrm{p}(Q). \tag{10}$$

Then, (9) and (10), together with our hypothesis, lead to

$$\mathrm{p}(K) \leq \mathrm{p}(Q) < \mathrm{p}(L) = \mathrm{p}(K),$$

which is a contradiction. □

2.2. **Spherical centrally symmetric convex bodies.** Next we consider symmetric convex bodies in the sphere.

A spherical convex body $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$ is said to be *centrally symmetric*, if there exists a point $c_K \in K$ such that for all $p \in \operatorname{bd} K$, $d(c_K, p) = d(c_K, p^*)$, where $p^* = \gamma_p^{c_K} \cap \operatorname{bd} K$, $p^* \neq p$ (see Figure 2).

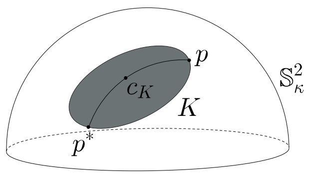

FIGURE 2. A centrally symmetric convex body K in $\mathbb{S}^2_\kappa$ with center c_K.

Next lemma shows that the point c_K is unique, which we call the *center* of K in $\mathbb{S}^2_\kappa$. The point p^* is the *symmetric point* of p with respect to c_K.

Lemma 2.3. *Let $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$ be centrally symmetric. Then the center c_K is unique.*

Proof. Let c_1, c_2 be two centers of K and for the sake of brevity we write $\gamma = \gamma_{c_1}^{c_2}$. Let $\{p,q\} = \gamma \cap \operatorname{bd} K$. Without loss of generality we may assume that the points p, c_1, c_2, q are "ordered on γ" as shown in Figure 3, i.e., if $x = \gamma(t_x)$, for $x \in \{p, q, c_1, c_2\}$, then $t_p < t_{c_1} < t_{c_2} < t_q$.

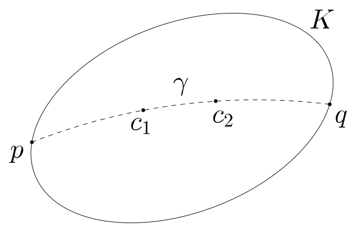

FIGURE 3. The center c_K of a centrally symmetric convex body $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$ is unique.

On the one hand, since c_1, c_2 are centers of K, we have that

$$d(p, c_1) = d(c_1, q) = d(p, c_2) = d(c_2, q) = \frac{d(p,q)}{2}.$$

On the other hand, since c_1, c_2, q lie on the same (minimizing) geodesic, then

$$d(c_1, q) = d(c_1, c_2) + d(c_2, q),$$

and thus we get $d(c_1, c_2) = 0$. Therefore $c_1 = c_2$. $\square$

We notice that if $K = D(p, r) \in \mathcal{K}(\mathbb{S}^2_\kappa)$ is a geodesic disc, then c_K is just the usual center p of K.

3. The spherical isodiametric inequality

We start this section showing Theorem 1.2, which establishes a Rosenthal-Szasz-type inequality in $\mathbb{S}^2_\kappa$ for centrally symmetric convex bodies. Then, Theorem 1.1 will be a direct consequence of the spherical isoperimetric inequality (4) and (6).

Proof of Theorem 1.2. First we show that

$$K \subseteq B_d\left(c_K, \frac{\mathrm{D}(K)}{2}\right). \tag{11}$$

Indeed, let $z \in K$ and we assume that $d(z, c_K) > \mathrm{D}(K)/2$. Then, denoting by z^* the symmetral of z with respect to c_K (see Figure 4), we have that

$$d(z, z^*) = d(z, c_K) + d(c_K, z^*) > \frac{\mathrm{D}(K)}{2} + \frac{\mathrm{D}(K)}{2} = \mathrm{D}(K),$$

which contradicts the fact that $\mathrm{D}(K)$ is the diameter of K. Thus $d(z, c_K) \leq \mathrm{D}(K)/2$, i.e., $z \in B_d\big(c_K, \mathrm{D}(K)/2\big)$.

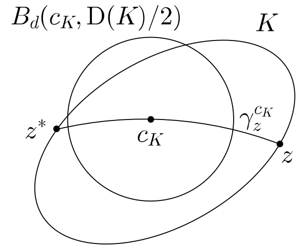

FIGURE 4. If K is symmetric then $K \subseteq B_d\big(c_K, \mathrm{D}(K)/2\big)$.

Then, Lemma 2.1 together with (8) yields

$$\mathrm{p}(K) \leq \mathrm{p}\Big(B_d\left(c_K, \tfrac{\mathrm{D}(K)}{2}\right)\Big) = \frac{2\pi}{\sqrt{\kappa}} \sin\left(\sqrt{\kappa}\frac{\mathrm{D}(K)}{2}\right).$$

Finally, if $\mathrm{p}(K) = \mathrm{p}\big(B_d\big(c_K, \mathrm{D}(K)/2\big)\big)$, since $K \subseteq B_d\big(c_K, \mathrm{D}(K)/2\big)$, then Lemma 2.2 ensures that $K = B_d\big(c_K, \mathrm{D}(K)/2\big)$ is a geodesic disc. $\square$

It is easy to check that when κ goes to 0 in (6), the classical inequality of Rosenthal and Szasz (3) is obtained.

We observe that if $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$ is centrally symmetric, then the relation $\mathrm{D}(K) = 2\mathrm{R}(K)$ holds. Indeed, from the inclusion (11) and the definition of circumradius, we immediately get $2\mathrm{R}(K) \leq \mathrm{D}(K)$. Moreover, for arbitrary $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$, there exists $p_0 \in \mathbb{S}^2_\kappa$ such that $K \subseteq B_d\big(p_0, \mathrm{R}(K)\big)$. Then, taking $p, q \in K \subseteq B_d\big(p_0, \mathrm{R}(K)\big)$, we obtain $d(p,q) \leq d(p,p_0) + d(p_0,q) \leq 2\mathrm{R}(K)$, and therefore, $\mathrm{D}(K) \leq 2\mathrm{R}(K)$.

We also notice that inequality (6) fails if the symmetry assumption is removed, as an octant of $\mathbb{S}^2_\kappa$ shows. In that case, the following proposition provides a (not sharp) bound for the perimeter in terms of the diameter.

Proposition 3.1. *Let $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$. Then*

$$\mathrm{p}(K) \leq \frac{4\pi}{\sqrt{3\kappa}} \sin\left(\sqrt{\kappa}\,\frac{\mathrm{D}(K)}{2}\right).$$

Proof. By definition of circumradius there holds $K \subseteq D\big(p, \mathrm{R}(K)\big)$ for some $p \in K$, and hence, using the monotonicity of the perimeter (see Lemma 2.1) and (8), we get

$$\mathrm{p}(K) \leq \mathrm{p}\big(D\big(p, \mathrm{R}(K)\big)\big) = \frac{2\pi}{\sqrt{\kappa}} \sin\big(\sqrt{\kappa}\,\mathrm{R}(K)\big).$$

Finally, we use the spherical Jung inequality

$$\mathrm{D}(K) \geq \frac{2}{\sqrt{\kappa}} \arcsin\left(\sqrt{\frac{n+1}{2n}} \sin\big(\sqrt{\kappa}\,\mathrm{R}(K)\big)\right),$$

for $n = 2$ (see [4, Theorem 2]) in order to bound the perimeter in terms of the diameter. $\square$

In the case of arbitrary (not necessarily centrally symmetric) convex bodies of the sphere, Proposition 3.1 and the spherical isoperimetric inequality (4) yield the following isodiametric relation for arbitrary convex bodies $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$:

Corollary 3.2. *Let $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$. Then*

$$\frac{16\pi^2}{3\kappa}\sin^2\left(\sqrt{\kappa}\,\frac{\mathrm{D}(K)}{2}\right) \geq 4\pi\mathrm{A}(K) - \kappa\mathrm{A}(K)^2.$$

We notice that inequalities in Proposition 3.1 and Corollary 3.2 are not sharp.

On the other hand, a series of inequalities of the form $\mathrm{p}(K)^2 - 4\pi\mathrm{A}(K) \geq F(K)$ where proved by Bonnesen during the 1920's, where $F(K)$ is a geometric non-negative functional, which vanishes only if K is a circle. Perhaps the most famous *Bonnesen inequality* is provided by the (planar, classical) circumradius and inradius:

$$\mathrm{p}(K)^2 - 4\pi\mathrm{A}(K) \geq \pi^2\big(\mathrm{R}(K) - \mathrm{r}(K)\big)^2. \tag{12}$$

Thus, using (3), Bonnesen's inequality (12) easily provides also a bound for the isodiametric deficit $\pi\mathrm{D}(K)^2 - 4\mathrm{A}(K)$, namely,

$$\pi\mathrm{D}(K)^2 - 4\mathrm{A}(K) \geq \pi\big(\mathrm{R}(K) - \mathrm{r}(K)\big)^2, \tag{13}$$

which we call a *Bonnesen-type isodiametric inequality*. Here equality also holds if and only if K is a circle.

In [10, Theorem 2.5], the following spherical Bonnesen (isoperimetric) inequality has been proved: for $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$ it holds

$$\begin{aligned} &\mathrm{p}(K)^2 - \big(4\pi\mathrm{A}(K) - \kappa\mathrm{A}(K)^2\big) \\ &\qquad \geq \frac{1}{4\kappa}\Big[\sin\big(\sqrt{\kappa}\,\mathrm{R}(K)\big) - \sin\big(\sqrt{\kappa}\,\mathrm{r}(K)\big)\Big]^2\big(2\pi - \kappa\mathrm{A}(K)\big)^2. \end{aligned} \tag{14}$$

Thus, using now Theorem 1.2, the spherical Bonnesen inequality (14) leads to a Bonnesen-type isodiametric inequality for centrally symmetric convex bodies in the sphere $\mathbb{S}^2_\kappa$:

Proposition 3.3. *Let $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$ be centrally symmetric. Then*

$$\begin{aligned} &\frac{4\pi^2}{\kappa}\sin^2\left(\sqrt{\kappa}\frac{\mathrm{D}(K)}{2}\right) - \big(4\pi\mathrm{A}(K) - \kappa\mathrm{A}(K)^2\big) \\ &\qquad \geq \frac{1}{4\kappa}\Big[\sin\big(\sqrt{\kappa}\,\mathrm{R}(K)\big) - \sin\big(\sqrt{\kappa}\,\mathrm{r}(K)\big)\Big]^2\big(2\pi - \kappa\mathrm{A}(K)\big)^2, \end{aligned}$$

with equality if and only if K is a geodesic disc.

We notice that $2\pi - \kappa\mathrm{A}(K) > 0$ for all $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$ because K is contained in an open hemisphere. Therefore, the spherical isodiametric inequality (Theorem 1.1) can be also obtained as a direct consequence of the above proposition.

4. Relating the perimeter and the area to the minimal width in the sphere

The width of a spherical convex body can be defined in the following way (see e.g. [6]). Let $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$ and let $p \in \operatorname{bd} K$ be a regular point, i.e., the (unit) tangent vector $\mathrm{v}(p)$ to $\operatorname{bd} K$ at p is unique. Let $\mathrm{n}(p) \in T_p\mathbb{S}^2_\kappa$ be the inner normal unit vector to the curve $\operatorname{bd} K$ at p, i.e., let

$$\mathrm{n}(p) = \frac{p \wedge \mathrm{v}(p)}{\bigl|p \wedge \mathrm{v}(p)\bigr|} = \sqrt{\kappa}\,\bigl[p \wedge \mathrm{v}(p)\bigr],$$

if it points to the interior of K, or $\sqrt{\kappa}\,\bigl[\mathrm{v}(p) \wedge p\bigr]$ on the contrary, and we take the geodesic $\gamma_{p,\mathrm{n}(p)}$ (see Figure 5).

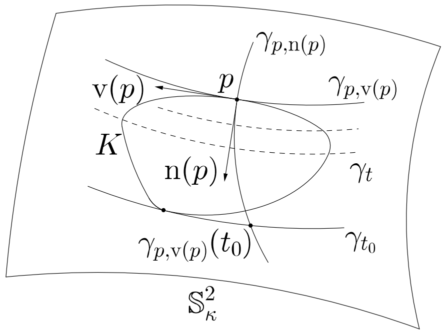

Figure 5. The width of a convex body K at $p \in \operatorname{bd} K$.

Next we consider the family γ_t of geodesics which are orthogonal to $\gamma_{p,\mathrm{n}(p)}$ at each point $\gamma_{p,\mathrm{n}(p)}(t)$, $t \geq 0$. Then, there exists the smallest positive number $t_0 > 0$ such that the geodesic γ_{t_0} passing through the point $\gamma_{p,\mathrm{n}(p)}(t_0)$ is tangent to $\operatorname{bd} K$, and clearly, $\gamma_{p,\mathrm{v}(p)}$ and γ_{t_0} are two 'supporting' geodesics to $\operatorname{bd} K$, orthogonal to $\gamma_{p,\mathrm{n}(p)}$, and such that K is contained in the strip determined by them.

The *width* $\omega(K,p)$ of K at $p \in \operatorname{bd} K$ is then defined as the length of $\gamma_{p,\mathrm{n}(p)}$ between the points $p = \gamma_{p,\mathrm{n}(p)}(0)$ and $\gamma_{p,\mathrm{n}(p)}(t_0)$, i.e.,

$$\omega(K,p) = d\left(p, \gamma_{p,\mathrm{n}(p)}(t_0)\right) = t_0.$$

If $p \in \operatorname{bd} K$ is not regular, we do the same construction for all tangent vectors $\mathrm{v}(p)$ to $\operatorname{bd} K$ at p, obtaining in this way a width-value $\omega\bigl(K,p,\mathrm{v}(p)\bigr)$ for each vector $\mathrm{v}(p)$; then we define the width $\omega(K,p)$ of K at $p \in \operatorname{bd} K$ as the minimum $\min_{\mathrm{v}(p)} \omega\bigl(K,p,\mathrm{v}(p)\bigr)$.

Finally the *minimal width* of K is defined as

$$\omega(K) = \min\bigl\{\omega(K,p) : p \in \operatorname{bd} K\bigr\}.$$

We would like to point out the similarity of this notion with the classical definition of minimal width of a planar (i.e., $\kappa = 0$) convex body (minimum distance between two supporting lines to K).

Next we show the relation between the perimeter/area and the minimal width.

Proof of Theorem 1.3. First we show the inclusion

$$B_d\left(c_K, \frac{\omega(K)}{2}\right) \subseteq K. \tag{15}$$

We reason by contradiction assuming that there exists a point

$$z \in B_d\left(c_K, \frac{\omega(K)}{2}\right) \backslash K.$$

Let $z_0 = \operatorname{bd} K \cap \overline{\gamma}_z^{c_K}$ and let $p \in \operatorname{bd} K$ be a boundary point of K such that $d(c_K, p)$ attains the minimum (see Figure 6).

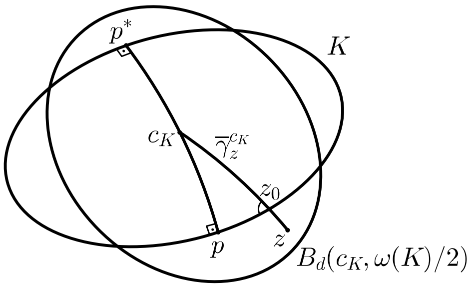

FIGURE 6. If K is symmetric then $K \supseteq B_d\big(c_K, \omega(K)/2\big)$.

Then (see, e.g., [11, p. 280, Corollary 26]) the geodesic $\gamma_p^{c_K}$ and the curve $\operatorname{bd} K$ are orthogonal at p, and thus

$$\omega(K,p) = d(p,p^*) = 2d(c_K,p) \leq 2d(c_K,z_0) < 2d(c_K,z) \leq \omega(K),$$

a contradiction.

Therefore $B_d\big(c_K, \omega(K)/2\big) \subseteq K$, and using the monotonicity of the area and the perimeter (see Lemma 2.1), together with (8), we obtain that

$$\mathrm{A}(K) \geq \mathrm{A}\Big(B_d\left(c_K, \tfrac{\omega(K)}{2}\right)\Big) = \frac{2\pi}{\kappa}\left[1 - \cos\left(\sqrt{\kappa}\,\frac{\omega(K)}{2}\right)\right] \quad \text{and}$$

$$\mathrm{p}(K) \geq \mathrm{p}\Big(B_d\left(c_K, \tfrac{\omega(K)}{2}\right)\Big) = \frac{2\pi}{\sqrt{\kappa}} \sin\left(\sqrt{\kappa}\frac{\omega(K)}{2}\right).$$

Finally, if the equality $\mathrm{p}(K) = \mathrm{p}\big(B_d(c_K, \omega(K)/2)\big)$ holds, or analogously, if $\mathrm{A}(K) = \mathrm{A}\big(B_d(c_K, \omega(K)/2)\big)$, since $K \supseteq B_d\big(c_K, \omega(K)/2\big)$, then Lemma 2.2 ensures that $K = B_d\big(c_K, \mathrm{D}(K)/2\big)$ is a geodesic disc. □

We notice that when κ tends to 0, the classical relation

$$\mathrm{p}(K) \geq \pi\omega(K)$$

of the plane is obtained (see e.g. [3, p. 84]). Moreover, the area inequality in (7) yields

$$\mathrm{A}(K) \geq \frac{\pi}{4}\omega(K)^2$$

if $\kappa \to 0$, a known lower bound for the ratio $A(K)/\omega(K)^2$ for the case of centrally symmetric sets (see e.g. [3, p. 83]).

We also observe that the expected relation $\omega(K) \leq \mathrm{D}(K)$ holds: indeed, from (15) and the definition of inradius, we immediately get $\omega(K) \leq 2\mathrm{r}(K)$, and thus

$$\omega(K) \leq 2\mathrm{r}(K) \leq 2\mathrm{R}(K) = \mathrm{D}(K).$$

5. Appendix: the hyperbolic case

Let $\mathbb{H}^2_\kappa$ be the 2-dimensional hyperbolic space of curvature $\kappa < 0$, for which we can consider, e.g., the Beltrami model. A subset $K \subsetneq \mathbb{H}^2_\kappa$ is said to be *convex* (in $\mathbb{H}^2_\kappa$) if for any $p, q \in K$, there exists a unique (arc length parametrized) geodesic segment joining p and q, which is contained in K. The set of all convex bodies in $\mathbb{H}^2_\kappa$ will be denoted by $\mathcal{K}(\mathbb{H}^2_\kappa)$. The other definitions are analogous to the ones in the sphere.

Now, the perimeter and the area of a geodesic disc in $\mathbb{H}^2_\kappa$ take the values

$$\begin{aligned}
\mathrm{p}\big(D(p,r)\big) &= \frac{2\pi}{\sqrt{-\kappa}} \sinh\left(\sqrt{-\kappa}\, r\right) \quad \text{and}\\
\mathrm{A}\big(D(p,r)\big) &= \frac{2\pi}{\kappa}\left[1 - \cosh\left(\sqrt{-\kappa}\, r\right)\right].
\end{aligned}$$

The monotonicity of the perimeter and the area (Lemma 2.1, see [10, Proposition 1.3]) with the equality cases (Lemma 2.2), as well as the uniqueness of the center of symmetry (Lemma 2.3) also hold. Thus, analogous proofs to the ones of the results in the sphere allow to show the following results:

Theorem 5.1 (Hyperbolic isodiametric inequality)**.** *Let $K \in \mathcal{K}(\mathbb{H}^2_\kappa)$ be centrally symmetric. Then*

$$-\frac{4\pi^2}{\kappa}\sinh^2\left(\sqrt{-\kappa}\frac{\mathrm{D}(K)}{2}\right) \geq 4\pi\mathrm{A}(K) - \kappa\mathrm{A}(K)^2,$$

and equality holds if and only if K is a geodesic disc.

Theorem 5.2 (Hyperbolic Rosenthal-Szasz theorem)**.** *Let $K \in \mathcal{K}(\mathbb{H}^2_\kappa)$ be centrally symmetric. Then*

$$\mathrm{p}(K) \leq \frac{2\pi}{\sqrt{-\kappa}}\sinh\left(\sqrt{-\kappa}\frac{\mathrm{D}(K)}{2}\right).$$

and equality holds if and only if K is a geodesic disc.

Theorem 5.3. *Let $K \in \mathcal{K}(\mathbb{H}^2_\kappa)$ be centrally symmetric. Then*

$$\mathrm{p}(K) \geq \frac{2\pi}{\sqrt{-\kappa}} \sinh\left(\sqrt{-\kappa}\,\frac{\omega(K)}{2}\right),$$

$$\mathrm{A}(K) \geq -\frac{2\pi}{\kappa}\left[\cosh\left(\sqrt{-\kappa}\,\frac{\omega(K)}{2}\right) - 1\right].$$

Equality holds in both inequalities if and only if K is a geodesic disc.

Proposition 5.4. *Let $K \in \mathcal{K}(\mathbb{H}^2_\kappa)$. Then*

$$\mathrm{p}(K) \leq \frac{4\pi}{\sqrt{-3\kappa}} \sinh\left(\sqrt{-\kappa}\,\frac{\mathrm{D}(K)}{2}\right).$$

Corollary 5.5. *Let $K \in \mathcal{K}(\mathbb{H}^2_\kappa)$. Then*

$$-\frac{16\pi^2}{3\kappa} \sinh^2\left(\sqrt{-\kappa}\,\frac{\mathrm{D}(K)}{2}\right) \geq 4\pi\mathrm{A}(K) - \kappa\mathrm{A}(K)^2.$$

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